

**Conjugacy criteria for half-linear ODE
in theory of PDE
with generalized p -Laplacian
and mixed powers**

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$$\begin{aligned} \operatorname{div} \left(A(x) \|\nabla y\|^{p-2} \nabla y \right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2} \nabla y \right\rangle \\ + c(x) |y|^{p-2} y + \sum_{i=1}^m c_i(x) |y|^{p_i-2} y = e(x), \end{aligned} \quad (\text{E})$$

- $x = (x_1, \dots, x_n)_{i=1}^n \in \mathbb{R}^n$, $p > 1$, $p_i > 1$,
- $A(x)$ is elliptic $n \times n$ matrix with differentiable components, $c(x)$ and $c_i(x)$ are Hölder continuous functions, $\vec{b}(x) = (b_1(x), \dots, b_n(x))$ is continuous n -vector function,
- $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)_{i=1}^n$ and $\operatorname{div} = \frac{\partial}{\partial x_1} + \dots + \frac{\partial}{\partial x_n}$ is are the usual nabla and divergence operators,
- q is a conjugate number to the number p , i.e., $q = \frac{p}{p-1}$,
- $\langle \cdot, \cdot \rangle$ is the usual scalar product in \mathbb{R}^n , $\|\cdot\|$ is the usual norm in \mathbb{R}^n , $\|A\| = \sup \{\|Ax\| : x \in \mathbb{R}^n \text{ with } \|x\| = 1\} = \lambda_{\max}$ is the spectral norm
- **solution** of (E) in $\Omega \subseteq \mathbb{R}^n$ is a differentiable function $u(x)$ such that $A(x) \|\nabla u(x)\|^{p-2} \nabla u(x)$ is also differentiable and u satisfies (E) in Ω
- $S(a) = \{x \in \mathbb{R}^n : \|x\| = a\}$,
 $\Omega(a) = \{x \in \mathbb{R}^n : a \leq \|x\|\}$,
 $\Omega(a, b) = \{x \in \mathbb{R}^n : a \leq \|x\| \leq b\}$

$$u'' + c(x)u = 0 \quad (1)$$

- Equation (1) is oscillatory if each solution has infinitely many zeros in $[x_0, \infty)$.
- Equation (1) is oscillatory if each solution has a zero $[a, \infty)$ for each a .
- Equation (1) is oscillatory if each solution has conjugate points on the interval $[a, \infty)$ for each a .
- All definition are equivalent (no accumulation of zeros and Sturm separation theorem).
- Equation is oscillatory if $c(x)$ is large enough. Many oscillation criteria are expressed in terms of the integral $\int^{\infty} c(x) dx$ (Hille and Nehari type)
- There are oscillation criteria which can detect oscillation even if $\int^{\infty} c(x) dx$ is extremely small. These criteria are in fact series of conjugacy criteria.

$$(p(t)u')' + c(t)u + \sum_{i=1}^m c_i(t)|u|^{\alpha_i} \operatorname{sgn} u = e(t) \quad (2)$$

where $\alpha_1 > \dots > \alpha_m > 1 > \alpha_{m+1} > \dots > \alpha_n > 0$.

Theorem A (Sun, Wong (2007)). *If for any $T \geq 0$ there exists a_1, b_1, a_2, b_2 such that $T \leq a_1 < b_1 \leq a_2 < b_2$ and*

$$\begin{cases} c_i(t) \geq 0 & t \in [a_1, b_1] \cup [a_2, b_2], \quad i = 1, 2, \dots, n \\ e(x) \leq 0 & t \in [a_1, b_1] \\ e(x) \geq 0 & t \in [a_2, b_2] \end{cases}$$

and there exists a continuously differentiable function $u(t)$ satisfying $u(a_i) = u(b_i) = 0, u(t) \neq 0$ on (a_i, b_i) and

$$\int_{a_i}^{b_i} \{p(t)u'^2(t) - Q(t)u^2(t)\} dt \leq 0 \quad (3)$$

for $i = 1, 2$, where

$$Q(t) = k_0|e(t)|^{\eta_0} \prod_{i=1}^m (c_i^{\eta_i}(t)) + c(t),$$

$$k_0 = \prod_{i=0}^m \eta_i^{-\eta_i} \text{ and } \eta_i, \quad i = 0, \dots, m \text{ are positive constants satisfying } \sum_{i=1}^m \alpha_i \eta_i = 1 \quad \text{and} \quad \sum_{i=0}^m \eta_i = 1,$$

then all solutions of (2) are oscillatory.

$$\Delta u + c(x)u = 0 \tag{4}$$

- Equation (4) is *oscillatory* if every solution has a zero on $\{x \in \mathbb{R}^n : \|x\| \geq a\}$ for each a .
- Equation (4) is *nodally oscillatory* if every solution has a nodal domain on $\{x \in \mathbb{R}^n : \|x\| \geq a\}$ for each a .
- Both definition are equivalent (Moss+Piepenbrink).

$$\operatorname{div}\left(\|\nabla u\|^{p-2} \nabla u\right) + c(x)|u|^{p-2}u = 0 \tag{5}$$

- Essentially the same approach to oscillation as in linear case
- The equivalence between two oscillations is open problem.

$$\begin{aligned} \operatorname{div} \left(A(x) \|\nabla y\|^{p-2} \nabla y \right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2} \nabla y \right\rangle \\ + c(x)|y|^{p-2}y + \sum_{i=1}^m c_i(x)|y|^{p_i-2}y = e(x), \end{aligned} \quad (\text{E})$$

DETECTION OF OSCILLATION FROM ODE

Theorem B (O. Došlý (2001)). *Equation*

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0 \quad (6)$$

is oscillatory, if the ordinary differential equation

$$\left(r^{n-1}|u'|^{p-2}u' \right)' + r^{n-1} \left(\frac{1}{\omega_n r^{n-1}} \int_{S(r)} c(x) \, dx \right) |u|^{p-2}u = 0 \quad (7)$$

is oscillatory. The number ω_n is the surface area of the unit sphere in \mathbb{R}^n .

J. Jaroš, T. Kusano and N. Yoshida proved independently similar result (for $A(x) = a(\|x\|)I$, $a(\cdot)$ differentiable).

OUR AIM

- Extend method used in Theorem A to (E). Derive a general result, like Theorem B.
- Derive a result which does depend on more general expression, than the mean value of $c(x)$ over spheres centered in the origin.
- Remove restrictions used by previous authors (for example Xu (2009) excluded the possibility $p_i > p$ for every i).

$$\begin{aligned} \operatorname{div} \left(A(x) \|\nabla y\|^{p-2} \nabla y \right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2} \nabla y \right\rangle \\ + c(x)|y|^{p-2}y + \sum_{i=1}^m c_i(x)|y|^{p_i-2}y = e(x), \end{aligned} \quad (\text{E})$$

MODUS OPERANDI

- Get rid of terms $\sum_{i=1}^m c_i(x)|y|^{p_i-2}y$ and $e(x)$ (join with $c(x)|y|^{p-2}y$) and convert the problem into

$$\operatorname{div} \left(A(x) \|\nabla y\|^{p-2} \nabla y \right) + \left\langle \vec{b}(x), \|\nabla y\|^{p-2} \nabla y \right\rangle + C(x)|y|^{p-2}y = 0.$$

- Derive Riccati type inequality in n variables.
- Derive Riccati type inequality in 1 variable.
- Use this inequality as a tool which transforms results from ODE to PDE.

Using generalized AG inequality $\sum \alpha_i \geq \prod \left(\frac{\alpha_i}{\eta_i} \right)^{\eta_i}$, if $\alpha_i \geq 0$, $\eta_i > 0$ and $\sum \eta_i = 1$ we eliminate the right-hand side and terms with mixed powers.

Lemma 1. Let either $y > 0$ and $e(x) \leq 0$ or $y < 0$ and $e(x) \geq 0$. Let $\eta_i > 0$ be numbers satisfying $\sum_{i=0}^m \eta_i = 1$ and $\eta_0 + \sum_{i=1}^m p_i \eta_i = p$ and let $c_i(x) \geq 0$ for every i . Then

$$\frac{1}{|y|^{p-2}y} \left(-e(x) + \sum_{i=1}^m c_i(x) |y|^{p_i-2}y \right) \geq C_1(x),$$

where

$$C_1(x) := \left| \frac{e(x)}{\eta_0} \right|^{\eta_0} \prod_{i=1}^m \left(\frac{c_i(x)}{\eta_i} \right)^{\eta_i}. \quad (8)$$

Remark: The numbers η_i from Lemma 1 exist, if $p_i > p$ for some i .

Lemma 2. Suppose $c_i(x) \geq 0$. Let $\eta_i > 0$ be numbers satisfying $\sum_{i=1}^m \eta_i = 1$ and $\sum_{i=1}^m p_i \eta_i = p$.

Then

$$\frac{1}{|y|^{p-2}y} \sum_{i=1}^m c_i(x) |y|^{p_i-2}y \geq C_2(x),$$

where

$$C_2(x) := \prod_{i=1}^m \left(\frac{c_i(x)}{\eta_i} \right)^{\eta_i} \quad (9)$$

Remark: The numbers η_i from Lemma 2 exist iff $p_i > p$ for some i and $p_j < p$ for some j .

Lemma 3. Let y be a solution of (E) which does not have zero on Ω . Suppose that there exists a function $C(x)$ such that

$$C(x) \leq c(x) + \sum_{i=1}^m c_i(x) |y|^{p_i-p} - \frac{e(x)}{|y|^{p-2}y}$$

Denote $\vec{w}(x) = A(x) \frac{\|\nabla y\|^{p-2} \nabla y}{|y|^{p-2}y}$. The function $\vec{w}(x)$ is well defined on Ω and satisfies the inequality

$$\operatorname{div} \vec{w} + (p-1)\Lambda(x) \|\vec{w}\|^q + \langle \vec{w}, A^{-1}(x)\vec{b}(x) \rangle + C(x) \leq 0 \quad (10)$$

where

$$\Lambda(x) = \begin{cases} \lambda_{\max}^{1-q}(x) & 1 < p \leq 2, \\ \lambda_{\min} \lambda_{\max}^{-q}(x) & p > 2. \end{cases} \quad (11)$$

Lemma 4. Let (10) hold. Let $l > 1$, $l^* = \frac{l}{l-1}$ be two mutually conjugate numbers and $\alpha \in C^1(\Omega, \mathbb{R}^+)$ be a smooth function positive on Ω . Then

$$\operatorname{div}(\alpha(x)\vec{w}) + (p-1) \frac{\Lambda(x)\alpha^{1-q}(x)}{l^*} \|\alpha(x)\vec{w}\|^q - \frac{l^{p-1}\alpha(x)}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p + \alpha(x)C(x) \leq 0$$

holds on Ω . If $\left\| A^{-1}\vec{b} - \frac{\nabla \alpha}{\alpha} \right\| \equiv 0$ on Ω , then this inequality holds with $l^* = 1$.

Theorem 1. Let the n -vector function \vec{w} satisfy inequality

$$\operatorname{div} \vec{w} + C_0(x) + (p-1)\Lambda_0(x) \|\vec{w}\|^q \leq 0$$

on $\Omega(a, b)$. Denote $\tilde{C}(r) = \int_{S(r)} C_0(x) \, d\sigma$ and $\tilde{R}(r) = \int_{S(r)} \Lambda_0^{1-p} \, d\sigma$. Then the half-linear ordinary differential equation

$$\left(\tilde{R}(r) |u'|^{p-2} u' \right)' + \tilde{C}(r) |u|^{p-2} u = 0, \quad ' = \frac{d}{dr}$$

is disconjugate on $[a, b]$ and it possesses solution which has no zero on $[a, b]$.

Theorem 2. Let $l > 1$. Let $l^* = 1$ if $\|\vec{b}\| \equiv 0$ and $l^* = \frac{l}{l-1}$ otherwise. Further, let $c_i(x) \geq 0$ for every i . Denote

$$\tilde{R}(r) = (l^*)^{p-1} \int_{S(r)} \Lambda^{1-p}(x) \, d\sigma$$

and

$$\tilde{C}(r) = \int_{S(r)} c(x) + C_1(x) - \frac{l^{p-1}}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x) \vec{b}(x) \right\|^p \, d\sigma,$$

where $\Lambda(x)$ is defined by (11) and $C_1(x)$ is defined by (8).

Suppose that the equation

$$\left(\tilde{R}(r) |u'|^{p-2} u' \right)' + \tilde{C}(r) |u|^{p-2} u = 0$$

has conjugate points on $[a, b]$.

If $e(x) \leq 0$ on $\Omega(a, b)$, then equation (E) has no positive solution on $\Omega(a, b)$.

If $e(x) \geq 0$ on $\Omega(a, b)$, then equation (E) has no negative solution on $\Omega(a, b)$.

Theorem 3 (non-radial variant of Theorem 2). Let $l > 1$ and let $\Omega \subset \Omega(a, b)$ be an open domain with piecewise smooth boundary such that $\text{meas}(\Omega \cap S(r)) \neq 0$ for every $r \in [a, b]$. Let $c_i(x) \geq 0$ on Ω for every i and let $\alpha(x)$ be a function which is positive and continuously differentiable on Ω and vanishes on the boundary and outside Ω . Let $l^* = 1$ if $\left\| A^{-1}\vec{b} - \frac{\nabla\alpha}{\alpha} \right\| \equiv 0$ on Ω and

$l^* = \frac{l}{l-1}$ otherwise. In the former case suppose also that the integral

$$\int_{S(r)} \frac{\alpha(x)}{\Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla\alpha(x)}{\alpha(x)} \right\|^p d\sigma$$

which may have singularity on $\partial\Omega$ if $\Omega \neq \Omega(a, b)$ is convergent for every $r \in [a, b]$. Denote

$$\tilde{R}(r) = (l^*)^{p-1} \int_{S(r)} \alpha(x) \Lambda^{1-p}(x) d\sigma$$

and

$$\tilde{C}(r) = \int_{S(r)} \alpha(x) \left(c(x) + C_1(x) - \frac{l^{p-1}}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x)\vec{b}(x) - \frac{\nabla\alpha(x)}{\alpha(x)} \right\|^p \right) d\sigma,$$

where $\Lambda(x)$ is defined by (11) and $C_1(x)$ is defined by (8) and suppose that equation

$$\left(\tilde{R}(r) |u'|^{p-2} u' \right)' + \tilde{C}(r) |u|^{p-2} u = 0$$

has conjugate points on $[a, b]$.

If $e(x) \leq 0$ on $\Omega(a, b)$, then equation (E) has no positive solution on $\Omega(a, b)$.

If $e(x) \geq 0$ on $\Omega(a, b)$, then equation (E) has no negative solution on $\Omega(a, b)$.

Theorem 4. Let l , Ω , $\alpha(x)$, $\Lambda(x)$ and $\tilde{R}(r)$ be defined as in Theorem 3 and let $c_i(x) \geq 0$ and $\mathbf{e}(x) \equiv \mathbf{0}$ on $\Omega(a, b)$. Denote

$$\tilde{C}(r) = \int_{S(r)} \alpha(x) \left(c(x) + C_2(x) - \frac{l^{p-1}}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x) \vec{b}(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p \right) d\sigma,$$

where $C_2(x)$ is defined by (9). If the equation

$$\left(\tilde{R}(r) |u'|^{p-2} u' \right)' + \tilde{C}(r) |u|^{p-2} u = 0$$

has conjugate points on $[a, b]$, then every solution of equation (E) has zero on $\Omega(a, b)$.

Similar theorems can be derived also for estimates of terms with mixed powers based on different methods than AG inequality (see R. M., Non-linear Analysis TMA 73 (2010)).

Conjugacy criteria for half-linear ODE in theory of PDE with generalized p -Laplacian and mixed powers

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$$\operatorname{div} \left(A(x) |\nabla u|^{p-2} \nabla u \right) + \tilde{c}(x) |\nabla u|^{q-2} \nabla u = c(x) + \sum_{i=1}^n c_i(x) |u|^{p_i-2} u + r(x) \quad (5)$$

- $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $p > 1$, $q > 1$.
- $A(x)$ is elliptic, c is a matrix with differentiable components, $c_i(x)$ and $c(x)$ are Hölder continuous functions, $\tilde{c}(x) = (\tilde{c}_1(x), \dots, \tilde{c}_n(x))$ is continuous n -vector function.
- $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right)^T$ and $\operatorname{div} \frac{\partial u}{\partial x_i} = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}$ is an usual scalar and divergence operators.
- q is a conjugate number to the number p , i.e., $\frac{1}{p} + \frac{1}{q} = 1$.
- (\cdot, \cdot) is the usual scalar product in \mathbb{R}^n , $|\cdot|$ is the usual norm in \mathbb{R}^n , $|A| = \sup \{ |Ax| : x \in \mathbb{R}^n \text{ with } |x| = 1 \} = \|A\|$ is the spectral norm.
- solution of (5) in $\Omega \subseteq \mathbb{R}^n$ is a differentiable function $u(x)$ such that $A(x) |\nabla u(x)|^{p-2} \nabla u(x)$ is also differentiable and a vector (5) in Ω .
- $R(x) = \{ x \in \mathbb{R}^n : |x| = 1 \}$.
- $\partial \Omega = \{ x \in \mathbb{R}^n : |x| = 1 \}$.
- $D(x, h) = \{ x \in \mathbb{R}^n : |x+h| \leq |h| \leq |x| \}$.

CONCEPT OF OSCILLATION FOR ODE

$$\operatorname{div} \left(A(x) |\nabla u|^{p-2} \nabla u \right) + \tilde{c}(x) |\nabla u|^{q-2} \nabla u = c(x) + \sum_{i=1}^n c_i(x) |u|^{p_i-2} u + r(x) \quad (6)$$

DETECTION OF OSCILLATION FROM ODE

Theorem 2 (D. Džur (2002)). Equation $\operatorname{div} \left(A(x) |\nabla u|^{p-2} \nabla u \right) + \tilde{c}(x) |\nabla u|^{q-2} \nabla u = c(x) |u|^{p-2} u = 0$ is oscillatory if the ordinary differential equation $\left(r'' + (r' - u^2) \left(\frac{1}{r} \right)' + \sum_{i=1}^n c_i(x) |u|^{p_i-2} u \right) u^{p-2} = 0$ is oscillatory. The number u is the surface area of the unit sphere in \mathbb{R}^n .

J. Jost, T. Kusano and N. Yoshida proved independently similar result (for $A(x) = x|x|^{-2}$, $c(x) = 0$ for every x).

Other uses

- Extended method used in Theorem 1 to (7). Derive a general result, the Theorem 5.
- Derive a result which does depend on more general expression, than the mean value of $r(x)$ over spheres centered in the origin.
- Remove restrictions used by previous authors (for example Xu (2008) excluded the possibility $p_i \neq p$ for every i).

CONCEPT OF OSCILLATION FOR ODE

$$\operatorname{div} \left(A(x) |\nabla u|^{p-2} \nabla u \right) + \tilde{c}(x) |\nabla u|^{q-2} \nabla u = c(x) + \sum_{i=1}^n c_i(x) |u|^{p_i-2} u + r(x) \quad (7)$$

Mixed variables

- Get rid of terms $\sum_{i=1}^n c_i(x) |u|^{p_i-2} u$ and $c(x)$ (gain with $c(x) |\nabla u|^{q-2} \nabla u$) and convert the problem into $\operatorname{div} \left(A(x) |\nabla u|^{p-2} \nabla u \right) + \tilde{c}(x) |\nabla u|^{q-2} \nabla u = c(x) |u|^{p-2} u = 0$.
- Derive Riccati type inequality in 1 variable.
- Use this inequality as a tool which transforms results from ODE to PDE.

Oscillation with mixed powers

$$\left(\frac{d}{dt} \right)^p u + \alpha(t) u + \sum_{i=1}^n \beta_i(t) |u|^{p_i-2} u = 0 \quad (8)$$

where $\alpha_1 > \dots > \alpha_n > 1 > \alpha_n > \dots > \alpha_1 > 0$.

Theorem 3 (Sun, Wong (2017)). If for any $T \geq 0$ there exists $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that $T \leq t < \beta_1 \leq \alpha_2 < \dots$ and $\begin{cases} \alpha(t) \geq 0 & t \in [\alpha_1, \beta_1], [\alpha_2, \beta_2], \dots \\ \alpha(t) \leq 0 & t \in [\beta_1, \alpha_2], [\beta_2, \alpha_3], \dots \end{cases}$

and there exists a continuously differentiable function $v(t)$ satisfying $v(t) = \alpha(t) > 0, v'(t) > 0$ on (α_1, β_1) and $\int_{\alpha_1}^{\beta_1} v(t) dt > 0$ and $\int_{\beta_1}^{\alpha_2} v(t) dt < 0$ for $i = 1, 2$, where $\Omega(t) = \alpha(t) \prod_{i=1}^n \beta_i(t) > 0$.

$\beta_1 = \prod_{i=1}^n \beta_i$ and $\alpha_1 = 0, \dots$ are positive constants satisfying $\sum_{i=1}^n \beta_i = 1$ and $\sum_{i=1}^n \alpha_i = 1$, then all solutions of (8) are oscillatory.

CONCEPT OF OSCILLATION FOR LINEAR PDE

$$\operatorname{div} \left(A(x) \nabla u \right) = 0 \quad (4)$$

- Equation (4) is oscillatory if every solution has a zero on $\{x \in \mathbb{R}^n : |x| \geq 1\}$ for each x .
- Equation (4) is nodally oscillatory if every solution has a nodal domain on $\{x \in \mathbb{R}^n : |x| \geq 1\}$ for each x .

Both definition are equivalent (Mařka-Popović).

CONCEPT OF OSCILLATION FOR HALF-LINEAR PDE

$$\operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) = 0 \quad (5)$$

- Establish the same approach to oscillation as in linear case.
- The equivalence between two oscillation is open problem.

CONCEPT OF OSCILLATION FOR LINEAR PDE

$$\operatorname{div} \left(A(x) \nabla u \right) = 0 \quad (4)$$

DETECTION OF OSCILLATION FROM ODE

Theorem 1 (D. Džur (2002)). Equation $\operatorname{div} \left(A(x) \nabla u \right) = c(x) |u|^{p-2} u = 0$ is oscillatory if the ordinary differential equation $\left(r'' + (r' - u^2) \left(\frac{1}{r} \right)' + \sum_{i=1}^n c_i(x) |u|^{p_i-2} u \right) u^{p-2} = 0$ is oscillatory. The number u is the surface area of the unit sphere in \mathbb{R}^n .

J. Jost, T. Kusano and N. Yoshida proved independently similar result (for $A(x) = x|x|^{-2}$, $c(x) = 0$ for every x).

Other uses

- Extended method used in Theorem 1 to (7). Derive a general result, the Theorem 5.
- Derive a result which does depend on more general expression, than the mean value of $r(x)$ over spheres centered in the origin.
- Remove restrictions used by previous authors (for example Xu (2008) excluded the possibility $p_i \neq p$ for every i).

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- Remove restrictions used by previous authors (for example Xu (2008) excluded the possibility $p_i \neq p$ for every i).

Lemma 3. Let v for a solution of (1) which does not have zero on Ω . Suppose that there exists a function $C(x)$ such that
$$C(x) \leq c(x) + \sum_{i=1}^n c_i(x) |u|^{p_i-2} u - \frac{c(x)}{p(x)}$$
 Define $\theta(x) = A(x) |\nabla u|^{p-2} \nabla u$. The function $\theta(x)$ is well defined on Ω and satisfies the inequality
$$\operatorname{div} \left((p-1)A(x) |\theta|^q + \left(A(x) \cdot \nabla \theta(x) \right) \cdot C(x) \right) \leq 0$$
 where $A(x) = \begin{cases} \frac{A(x)}{p(x)} & 1 < p \leq 2 \\ \frac{A(x)}{p(x)-1} & p > 2 \end{cases}$

Lemma 4. Let (3) hold. Let $t > 1$, $t^p = 1$ be two mutually conjugate numbers and $a \in C^1(\mathbb{R}, \mathbb{R}^n)$ be a smooth function positive on \mathbb{R} . Then
$$\operatorname{div} \left(A(x) |\nabla u|^{p-2} \nabla u \right) + \tilde{c}(x) |\nabla u|^{q-2} \nabla u = c(x) + \sum_{i=1}^n c_i(x) |u|^{p_i-2} u + r(x) \leq 0$$
 holds on Ω . If $A(x) \cdot \nabla \frac{a(x)}{p(x)} \geq 0$ on Ω , then this inequality holds with $t^p = 1$.

Theorem 3. Let the n -vector function θ satisfy inequality
$$\operatorname{div} \left(C(x) \theta + (p-1)A(x) |\theta|^q \right) \leq 0$$
 on $\Omega(x, h)$. Denote $C(x) = \int_{\Omega(x, h)} C(x) dx$ and $R(x) = \int_{\Omega(x, h)} A(x) dx$. Then the half-linear ordinary differential equation
$$\left(R(x) |r'|^{p-2} r' \right)' + C(x) |r|^{p-2} r = 0, \quad r' = \frac{d}{dt} r$$
 is dissipative on $[a, b]$ and it possesses solution which has no zero on $[a, b]$.

Theorem 4. Let $t > 1$. Let $t^p = 1$ and $t^q = 1$ be mutually conjugate. Further, let $\theta(x) \geq 0$ for every x . Denote
$$R(x) = \left(t^p - 1 \right) \int_{\Omega(x, h)} A(x) |\nabla u|^{p-2} u dx$$
 and
$$C(x) = \int_{\Omega(x, h)} c(x) + C_1(x) - \frac{p^p}{p^p - 1} \left(A(x) \cdot \nabla \theta(x) \right) dx$$
 where $A(x)$ is defined by (1) and $C_1(x)$ is defined by (3). Suppose that the equation
$$\left(R(x) |r'|^{p-2} r' \right)' + C(x) |r|^{p-2} r = 0$$
 has conjugate points on $[a, b]$. If $\theta(x) \geq 0$ on $\Omega(x, h)$, then equation (1) has no positive solution on $\Omega(x, h)$. If $\theta(x) \geq 0$ on $\Omega(x, h)$, then equation (1) has no negative solution on $\Omega(x, h)$.

Theorem 3 (non-radial variant of Theorem 3). Let $t > 1$ and let $\Omega \subset \Omega(x, h)$ be an open domain with piecewise smooth boundary such that $\operatorname{meas}(\Omega) > 0$ for every $x \in [a, b]$. Let $\theta(x) \geq 0$ on Ω for every x and let $\theta(x)$ for a function which is positive and continuously differentiable on Ω and vanishes on the boundary and inside Ω . Let $t^p = 1$ and $t^q = 1$ be mutually conjugate and $\theta(x) \geq 0$ on Ω and $\theta(x) \geq 0$ otherwise. In the former case suppose also that the integral
$$\int_{\Omega(x, h)} \frac{A(x)}{p(x)} dx > 0, \int_{\Omega(x, h)} A(x) |\nabla u|^{p-2} u dx > 0$$
 has conjugate points on $[a, b]$ and $R(x) = \left(t^p - 1 \right) \int_{\Omega(x, h)} A(x) |\nabla u|^{p-2} u dx$ and
$$C(x) = \int_{\Omega(x, h)} c(x) + C_1(x) - \frac{p^p}{p^p - 1} \left(A(x) \cdot \nabla \theta(x) \right) dx$$
 where $A(x)$ is defined by (1) and $C_1(x)$ is defined by (3) and suppose that equation
$$\left(R(x) |r'|^{p-2} r' \right)' + C(x) |r|^{p-2} r = 0$$
 has conjugate points on $[a, b]$. If $\theta(x) \geq 0$ on $\Omega(x, h)$, then equation (1) has no positive solution on $\Omega(x, h)$. If $\theta(x) \geq 0$ on $\Omega(x, h)$, then equation (1) has no negative solution on $\Omega(x, h)$.

Theorem 4. Let $C_1(x)$, $A(x)$ and $R(x)$ be defined as in Theorem 3 and let $c(x) \geq 0$ and $\theta(x) \in \Omega(x, h)$. Denote
$$C(x) = \int_{\Omega(x, h)} c(x) + C_1(x) - \frac{p^p}{p^p - 1} \left(A(x) \cdot \nabla \theta(x) \right) dx$$
 where $C_1(x)$ is defined by (3). If the equation
$$\left(R(x) |r'|^{p-2} r' \right)' + C(x) |r|^{p-2} r = 0$$
 has zero on $\Omega(x, h)$.

Similar theorems can be derived also for estimates of terms with mixed powers instead of the mixed methods than AG inequality (see M. R. Niebur, Nonlinear Analysis, TMA 11 (1982)).